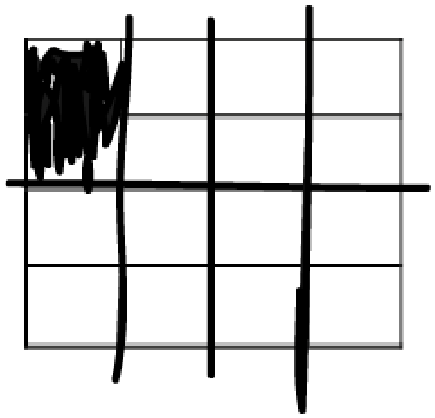
Answer Key Chapter 1: Developing Understanding of Fractions Through Visual Models

1. Compare the questions in Figures 1.31 and 1.32. Data from the OGAP Exploratory Study (OGAP, 2005) showed that students had a more difficult time with the question in Figure 1.31 than with the question in Figure 1.32. Provide a possible explanation for why one is more difficult than the other.

Sample Answer

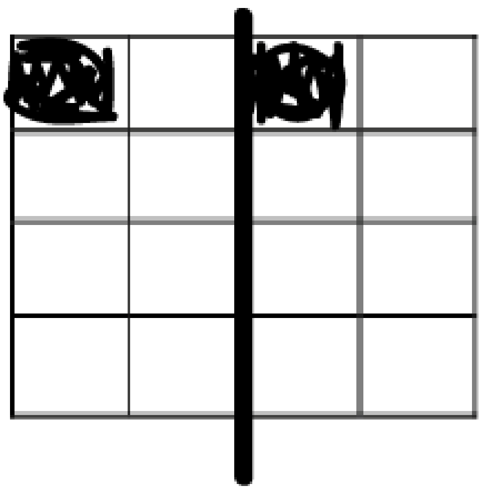
In both questions, the number of parts in the whole is 16, which is a multiple of the denominator 8. The equation related to the diagram in Figure 1.31 is $4 \times 4 = 16$ (4×4 array), and the equation related to the diagram in Figure 1.32 is $8 \times 2 = 16$ (2×8 array). For the given fraction $\frac{1}{8}$, there is a difference in recognizing how the denominator, 8, appears in the two diagrams. A student's eyes will have a much easier time spotting 8 equal parts (8 columns) in the second diagram. Finding $\frac{1}{8}$ of the diagram in Figure 1.31 involves dividing the 16 parts into eight equal parts without the advantage of the number of columns or rows equaling the magnitude of the denominator. This requires partitioning, as shown in Answer Key Figure 1.1.

Answer Key Figure 1.1. Finding $\frac{1}{8}$ of the figure by dividing it into eight equal parts.



Students who do not partition the figure into eight equal parts may arrive at an incorrect solution by applying inappropriate whole number reasoning and shade only 1 part or 8 parts. Other students may determine the correct answer using an out-of-parts strategy, as shown in Answer Key Figure 1.2. (See Chapter 3 for discussion of the out-of-parts strategy.) Although the answer is correct, students who only solve part to whole problems using an out-of parts strategy may encounter difficulties as the number of parts in the whole become greater or as the fractions become more complex (e.g., find $\frac{1}{8}$ of \$128).

Answer Key Figure 1.2. Finding $\frac{1}{8}$ of the figure by finding one out of eight pieces two times.



2. **Why do you think that it is more difficult for a student to determine the fractional part of a whole when the number of parts in the whole is a factor or multiple of the denominator rather than when the number of parts in the whole is equal to the denominator?**

Sample Answer

First, look at a case in which the number of parts in the whole is equal to the magnitude of denominator in the given fraction. In this situation, students have the easiest time responding with a correct answer. Because the denominator of the fraction matches the number of parts in the whole, the answer will be correct if the student selects a number of parts in the whole that matches the numerator of the given fraction. In fact, the student will respond correctly even if she didn't identify the crucial information that the denominator of the fraction matches the number of parts in the whole but simply selected the number of parts in the model that matched the numerator of the fraction.

Next, consider a case in which the number of parts in the model is a multiple of the denominator of the given fraction. How difficult this case will be for a student varies with the denominator of the fraction and how easy or difficult it is for the student to spot this denominator in the given model. That is exactly the situation that was discussed in the previous problem and it requires an understanding of partitioning.

Finally, we consider a case in which the number of parts in the model is a factor of the denominator of the fraction (see Answer Key Figure 1.3). This situation can present several challenges for students. The first challenge relates to the student's ability to reason multiplicatively. This idea will be presented through two tasks, each involving the fraction $\frac{1}{6}$.

Answer Key Figure 1.3. Shade $\frac{1}{6}$ in each figure (A and B).

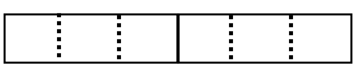
A B

In each of these models, the student must find a way to partition (i.e., divide) both figures into six sections with equal area. Both tasks require the recognition that the six sections can be created by adding to the lines given in the figures.

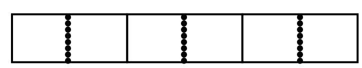
In figure B, each section can be divided into two parts, resulting in six sections of the same size. This contrasts with figure A, in which each section must be cut into three sections of the same size (see Answer Key Figure 1.4).

Answer Key Figure 1.4. Partitioning a figure where the number of parts in the whole is a factor of the denominator.

A



B



It is often easier for students to partition an object into two pieces of the same size than to divide it into three pieces of the same size. Therefore, it may be easier for many students to mark $\frac{1}{6}$ in figure B than in figure A.

3. **You have just completed the first part of your fraction unit with your third grade students. Up to this point your students have been finding the fractional part of an area as in Figure 1.33. Students have been very successful with questions like these. Today you are going to ask your students to find $\frac{3}{4}$ of the objects in a bag. There are**

four marbles and eight buttons in the bag. What aspects of the task may cause problems for your students? Explain.

Sample Answer

There are aspects of this task that may cause difficulties even though students have been successful in solving problems that involve finding fractional parts of areas. In the problem shown in Figure 1.33, students need to partition the region into four equal parts and shade three of those parts. The bag problem, however, requires an understanding of the features of sets.

a) If students are unfamiliar with working with sets of objects, they might be unsure what to do given objects that are not the same size, color, or type.

b) Because the number of objects in the bag is a multiple of the denominator, the problem requires an understanding of division and multiplication. Students need to divide the set of 12 objects into four parts, each with three objects. Each group of three objects represents $\frac{1}{4}$ of the objects in the bag. The last part of the task requires students to understand that $\frac{3}{4}$ of the objects is three times greater than $\frac{1}{4}$ of the objects.

Answer Key Chapter 2: Fractions Are Quantities

1. Earlier in the chapter you reviewed Michael's pre-assessment response (Figure 2.1) to placing $\frac{1}{3}$ and $\frac{1}{4}$ on a number line from 0 to 1. His pre-assessment response (Figure 2.10) is shown again below with his post- assessment response to the same question (Figure 2.11). Review both responses and then answer the questions below.

(a) What evidence in Michael's pre-assessment response suggests that Michael inappropriately used whole number reasoning when placing $\frac{1}{3}$ and $\frac{1}{4}$ on the number line?

Sample Answer

In Figure 2.10, Michael states that "$\frac{1}{3}$ comes after $\frac{1}{2}$, and then $\frac{1}{4}$ after $\frac{1}{3}$ because it goes 1, 2, 3, 4." Not only has Michael placed the fractions in the same order as the magnitude of their denominators are ordered, he appears to have spaced these fractions equally along the number line.

(b) What was Michael able to do on the post-assessment that was not shown in his response on the pre-assessment?

Sample Answer

On the post-assessment (Figure 2.11), Michael shows an understanding of equivalence of some fractions, He states that $\frac{4}{4} = \frac{3}{3} = 1$ and that $\frac{2}{4} = \frac{1}{2}$. He also correctly locates $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, and $\frac{4}{4}$ as well as $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{3}{3}$ on the number line. Michael provides a clear description of his strategy for locating these fractions on the number line. None of this understanding appears in his pre-assessment solution.

(c) Michael's post-assessment response is very different from his pre-assessment response. What is one instructional focus that might have helped Michael move from inappropriate use of whole number reasoning to treating each fraction as a single quantity? Look at Michael's post-assessment response in relation to the *OGAP Fraction Progression* to support your answer.

Sample Answer

Michael's post-assessment strategy (Figure 2.11) includes his description that he "split them in 3rds," which matches his locations of $\frac{1}{3}$ and $\frac{2}{3}$ at points that divide the unit (0–1) into three sections equal length. In addition, his description that he "split them in half and knew that was $\frac{1}{4}$" is hard to interpret, but could refer to an effective strategy for partitioning a section into fourths: first, split the section into halves, and then split each of the halves into halves. He indicates that his strategy leads him to the location of $\frac{1}{4}$ (which is correctly marked). Michael's post-assessment response (Figure 2.11), showing correct locations of fractions on the number line, shows evidence that he treated each fraction as a single quantity. In the same way that 0 and 1 correspond to particular points on the number line, he marked fractions at particular points on the number line, setting up a correspondence between one fraction and one point. These strategies for accurately partitioning and locating fractions on a number line are examples of *Fractional Strategies* exemplified by accurately locating fractions on a number line and comparing and ordering fractions using a range of strategies. Michael's response leads one to believe that he had additional instruction using number lines and with partitioning. For more detail about number lines and partitioning, see Chapter 6, Number Lines and Fractions, and Chapter 4, Equipartitioning.

2. Figures 2.12 and 2.13 include Mark's and Kim's responses to a question about the magnitude of a fraction. Both responses include diagrams generated by the students. Consider their responses, and then answer the following questions.

(a) What is Mark able to do? What is the evidence in Mark's response that leads you to believe that his ability to compare $\frac{3}{5}$ to a benchmark fraction is still developing, but is fragile and easily destabilized? Explain.

Sample Answer

Mark has drawn a serviceable area model (Figure 2.12) that includes a representation of $\frac{3}{5}$. One might expect that Mark would consider the choices presented in the problem and select B, $\frac{1}{2}$. It appears that Mark made that selection, but then replaced that choice with C, 5.

Mark's selection of 5 seems to be based on the denominator of the fraction $\frac{3}{5}$, leading one to speculate that Mark is not treating the fraction as a single quantity. Mark's response is indicative of a fragile and developing understanding. While he drew an area model to represent $\frac{3}{5}$, his erasure of $\frac{1}{2}$ (B) and selection of 5 (C) is evidence that he is uncertain how his model relates to the magnitude of $\frac{3}{5}$.

(b) What was Kim able to do? What is the evidence that Kim is using sound fraction reasoning? Explain.

Sample Answer

Kim (Figure 2.13) was able to accurately draw area models for $\frac{1}{2}$ and $\frac{3}{5}$. She also found the difference between $\frac{3}{5}$ and $\frac{1}{2}$, 5, and 8.

3. Review Kim's solution (Figure 2.11) one more time. Kim included very carefully drawn and accurate area models for $\frac{1}{2}$ and $\frac{3}{5}$. To what extent did Kim's explanation require these area models?

Sample Answer

Kim's written explanation suggests that she is comfortable with reasoning about the magnitude of fractions and with subtraction of fractions and may not have needed area model representations.

4. Review the evidence in Willy's response found in Figure 2.14, and answer the questions that follow.

(a) What is the evidence in Willy's response that he had sound fractional reasoning?

Sample Answer

Willy considered an answer from the four given choices of 20, 8, $\frac{1}{2}$, and 1. His explanation that "$\frac{7}{8}$ is almost 1" helps him close in on the correct answer. Willy also indicates that the sum of $\frac{7}{8}+\frac{1}{12}$ "is just going to be a little less than 1." In order to select the correct answer, Willy made a sound approximation of both the magnitude of $\frac{7}{8}$ and the magnitude of $\frac{7}{8}+\frac{1}{12}$.

(b) If Willy had the time to rewrite his response, how might his sentence be rewritten to clarify what you think Willy had in mind?

Sample Answer

Willy might have written, "The sum of $\frac{7}{8}+\frac{1}{12}$ is closest to 1 because $\frac{7}{8}$ is one eighth less than one, and one-twelfth is less than one-eighth. This means that the sum of $\frac{7}{8}+\frac{1}{12}$ is slightly less than 1."

(c) Do you think that Willy decided that the sum "is just going to be a little less than 1" without computing the sum? If he didn't add the fractions, what reasoning do you think Willy used to decide that the sum "is just going to be a little less than 1?"

Sample Answer

There is no evidence in Willy's explanation that he found the exact sum of $\frac{7}{8}+\frac{1}{12}$. Without finding the sum of the two fractions, one can select the correct answer through knowledge of two facts: first, $\frac{7}{8}+\frac{1}{8}=1$, and second, $\frac{1}{12}<\frac{1}{8}$. Putting these facts together could have led Willy to his explanation that the sum "is just going to be a little less than 1." This type of reasoning can be developed through the use of models.

Answer Key Chapter 3: What Is the Whole?

1. **Explain how the lesson learned from the candy bar vignette was applied by the students in the following vignette.**

    A group of fourth-grade students compared $\frac{5}{8}$ and $\frac{2}{3}$. As the students were presenting and discussing their solutions, one student said that it didn't really matter which was bigger because $\frac{2}{3}$ is only $\frac{1}{24}$ bigger than $\frac{5}{8}$, and that wasn't very big. Another student immediately piped up and said that it depends upon the size of the whole. If the whole is really big, than $\frac{1}{24}$ could be really big **(OGAP, 2005).**

Sample Answer

The candy bar vignette illustrates the importance of associating a fraction with a unit. The second student understood that a fraction without an associated unit is insufficient. In this situation, the second student was concerned about the size of the unit. In other cases, one might be concerned about the meaning of the unit.

For example, suppose you receive a call from a friend traveling to visit your house. You ask how far she has traveled and get the answer, "about two-thirds." Here, two-thirds mentioned without a unit could mean two-thirds of an hour, two-thirds of the trip or, two-thirds of a mile. You may know that the trip takes about one and a half hours, but without a unit mentioned, it will be impossible to approximate your friend's arrival time.

2. **It was suggested that Bob (Figure 3.23) may have used the same strategy to solve parts A and B of the candy bar problem. In both cases, Bob added the number of pieces that resulted in a candy bar with the total number of pieces equal to the denominator of the fraction given. Although this method resulted in a correct response on part A, the question remains, did Bob use inappropriate whole number reasoning to solve both questions? What questions might you ask to determine if Bob is using inappropriate whole number reasoning and to help Bob deepen his understanding of finding the whole when given a fractional part?**

Sample Answer

One might start with questions that help determine Bob's underlying understandings related to solving the $\frac{1}{5}$ of a candy bar problem and the $\frac{7}{8}$ of a candy bar problem.

(a) Explain how you determined the size of the whole candy bar when you were given $\frac{1}{5}$.

(b) Explain how you determined the size of the whole candy bar when you were given $\frac{7}{8}$.

You may find that Bob used whole number reasoning (he added the number of pieces needed to equal the magnitude of the denominator), or that Bob applied appropriate reasoning when given $\frac{1}{5}$ of a candy bar, but not when considering $\frac{7}{8}$ of a candy bar.

To help students develop a strategy for solving the $\frac{7}{8}$ problem, one might start with a series of easier "non-unit" fractions, such as $\frac{3}{4}$ of a candy bar. Ask questions such as:

- You have $\frac{3}{4}$ of a candy bar. Is that more or less than $\frac{1}{2}$ of a candy bar? More or less than a whole candy bar? Explain.

- How close is this to a whole candy bar? Explain.
- How can you determine the size of the piece that, when added to $\frac{3}{4}$ of a candy bar, equals a whole candy bar?

As students begin to understand the underlying concepts, ask them to solve similar questions with sets of objects and with a variety of fractions, such as $\frac{2}{3}$, that are one part away from the whole. Challenge students with similar problems that include fractions that are two parts from the whole, such as $\frac{3}{5}$, or three parts from the whole, such as $\frac{5}{8}$.

3. Earlier in the chapter we examined part of Matt's work on the "fractions of a square" problem. Shown in Figure 3.26 is all of Matt's written work, in which Matt indicated that the sum of all the parts is $\frac{20}{80}$. Matt's teacher asked him to describe this part of his solution.

He explained: "Twenty-eightieths. 20 + 80 = 100, that's the whole!"

Use the evidence in the student work (Figure 3.26) to answer the following questions.

(a) What understandings of fractional parts of an area model are evidenced in Matt's response? Describe the evidence.

Sample Answer

Matt successfully found the fractional part of the whole for parts A, B, C, and E. He placed the correct fraction in these sections.

(b) What errors are evidenced in Matt's response? Describe the evidence.

Sample Answers

- When considering sections D, F, G, H, and I, it appears that Matt considered one-fourth of the large square as the whole.
- Matt used inappropriate whole number reasoning when adding the nine fractions (he added the numerators to get a sum of 20 and he added the denominator to get a sum of 80).

(c) What potential questions might you ask Matt that would help him focus on identifying the whole?

Sample Answer

Ask Matt to describe the fractional value of each piece and the corresponding whole (e.g., "section A is $\frac{1}{8}$ of the large square"). This might help Matt see that in some instances he used the large outer square as the whole and in other cases he used one-fourth of the large square as the whole. Ask Matt to compare section C (or E) to section D. How much larger is D than C? This would help Matt focus on the relative magnitude of the pieces. Hopefully Matt would see that section D is twice as large as section C, and if C is $\frac{1}{16}$, D must be $\frac{2}{16}$ or $\frac{1}{8}$.

(d) What potential questions might you ask to help Matt rethink his conclusion that 20 + 80 = 100, and that is the whole? Provide a rationale for each question.

Sample Answers

- Ask Matt to consider the sum of sections F, G, H, and I (using the context of money may be helpful). Matt might realize that the sum of these four pieces is greater than one whole, so $\frac{20}{80}$ cannot equal one whole.
- Ask Matt to shade $\frac{20}{80}$ of a 10 × 8 grid. Ask Matt if he shaded the whole grid. Ask him to explain any discrepancies between his answer to the problem and shading $\frac{20}{80}$ of a 10 × 8 grid.

4. Read Tiara's and Maggie's responses to the "candies in a dish" problem in Figures 3.27 and 3.28. Although both students successfully answered the question, they each used a different strategy. Explain how an understanding of the whole is reflected in each of their solutions.

Sample Answer

The problem involved comparing fractional amounts with the same size whole (candies in the same dish). Tiara selected the number 10 to represent the whole (the total number of candies in the dish). She used this number when considering both $\frac{2}{5}$ and $\frac{3}{10}$ of the candies. Maggie used two area models that are the same size to compare the two fractions. Even though Tiara used a number (10) and Maggie used two area models to represent the whole, they both seem to understand that the same size whole must be used when comparing the fractions. One might wonder if Tiara selected 10 because it was equal to the magnitude of one of the denominators, or if Tiara understood that there could be more than 10 candies in the dish.

5. **Review Jayden's work in Figure 3.29 and answer the following questions.**

(a) **Jayden's solution does not reflect the situation. What is the evidence?**

(b) **What are some questions that you could ask that may help Jayden rethink her solution?**

Sample Answer

The key to solving this problem is an understanding that the context of the problem requires two different-sized wholes. One must also understand that $\frac{3}{5}$ is greater than $\frac{1}{2}$ when related to the same size whole (e.g., two classrooms, each with the same number of students), but may not be when associated with any two classrooms that probably differ in their total number of students. To help Jayden understand this idea, one might start by asking about the number of students in different classrooms in her school. Jayden should realize that in this context, one is not comparing wholes of the same size. Once she understands this, you might ask:

- In this situation, which classroom must have the most students and why? (Mr. Taylor's class because $\frac{1}{2} < \frac{3}{5}$, so a greater number of students is needed.)
- Experiment with various numbers of students in the two classes until you find a situation in which $\frac{1}{2}$ of Mr. Taylor's class is a greater number than $\frac{3}{5}$ of Mrs. Smith's class.

Answer Key Chapter 4: Equipartitioning

1. **Study Mandy's and Mark's responses in Figures 4.27 and 4.28 and then answer the following questions.**

   (a) **What strategy does Mandy use to place $\frac{1}{4}$ on the number line? Does she use the same strategy to place $\frac{1}{3}$ on the number line? Explain, using evidence from Mandy's response.**

Sample Answer

Mandy used a halving strategy to place $\frac{1}{4}$ on the number line. She explained, "half of $\frac{1}{2}$ is $\frac{1}{4}$."

Her placement of $\frac{1}{4}$ on the number line is relatively accurate.

Mandy's strategy for locating $\frac{1}{3}$ is not so clear. Mandy did place $\frac{1}{3}$ between the points on the number line marked $\frac{1}{4}$ and $\frac{1}{2}$. Her explanation that she "just put $\frac{1}{3}$ on after it" does not clarify her thinking. Mandy's teacher might ask Mandy to more clearly explain her placement of $\frac{1}{3}$.

   (b) **What strategy does Mark use to place $\frac{2}{3}$, $\frac{8}{12}$, and $\frac{8}{3}$ on the number line? Explain, using evidence from Mark's work.**

Sample Answer

Mark used equipartitioning to accurately locate all three fractions on the number line. Notice that his partitioning between 0 and 1 clearly shows both twelfths and thirds. Mark probably used his understanding that $2=\frac{6}{3}$ to help him locate $\frac{8}{3}$ on the number line. Because of this, he needed to equipartition only the thirds between 2 and 3. Thus, eight-thirds is equal to $2+\frac{2}{3}$.

(c) Because this is the only evidence that you have about each student's level of equipartitioning, what else might you want to know to determine the next instructional steps?

Sample Answer

You might want to know if Mandy can locate and label $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ on a 0 to 1 number line. You might then ask Mandy to locate $\frac{1}{8}$ on the number line. This will clarify whether Mandy has an understanding of the next level of the halving strategy. Following that, you give Mandy the same number line with 0 and 1 labeled and ask her to locate both $\frac{1}{3}$ and $\frac{2}{3}$. Mandy should have a clear understand of this partitioning before continuing her lessons on fractions and their locations on the number line.

Mark could be asked to locate other proper, improper, and mixed numbers on number lines. You might vary the magnitude and equipartitioning of the number lines presented to Mark. Mark appears to be ready to use the number line to consider addition and subtraction of fractions questions such as:

- How much greater is fraction A than fraction B?
- How much less is fraction B than fraction A?

2. John (Figure 4.29) and Kim (Figure 4.30) answered different problems that involve dividing into "fair shares." Study their responses, and then answer the following questions.

(a) John and Kim both used partitioning in their solutions. How are their strategies different? Explain.

Sample Answer

John's solution treats the unit as the entire order of four pizzas, with 12 students sharing the order. Each student will get $\frac{1}{12}$ of the order. John's diagram shows the unit (the four pizzas) partitioned into 12 parts of equal area, with each of the 12 parts labeled. John explains that each student's share is $\frac{1}{3}$ of a pizza.

Kim's solution treats the unit as one piece of construction paper, not three pieces. Each student's fair share is determined by giving each student a fair share of each piece of construction paper. Kim has partitioned each piece of construction paper, accurately drawing and labeling parts 1 through 6 for the six students. Kim, however, has not included in her response a fraction that describes each student's total share. One would want Kim to understand that each student receives $\frac{3}{6}$ or $\frac{1}{2}$ of a piece of construction paper.

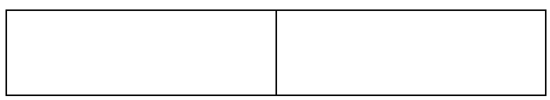
(b) What activity or question might help Kim partition each piece of paper into halves instead of sixths and to recognize that each student receives $\frac{1}{2}$ of a piece of construction paper?

Sample Answer

One possible activity for Kim has two parts:

- Two students are sitting at a table. The teacher gives them one piece of construction paper to fairly share between them. How much construction paper does each student get (Answer Key Figure 4.1)?

Answer Key Figure 4.1. One piece of construction paper shared by two students. Each student gets one-half of a piece of construction paper.



- Four students are sitting at the next table. The teacher gives them two pieces of construction paper to fairly share between them. How much construction paper does each student get (Answer Key Figure 4.2)?

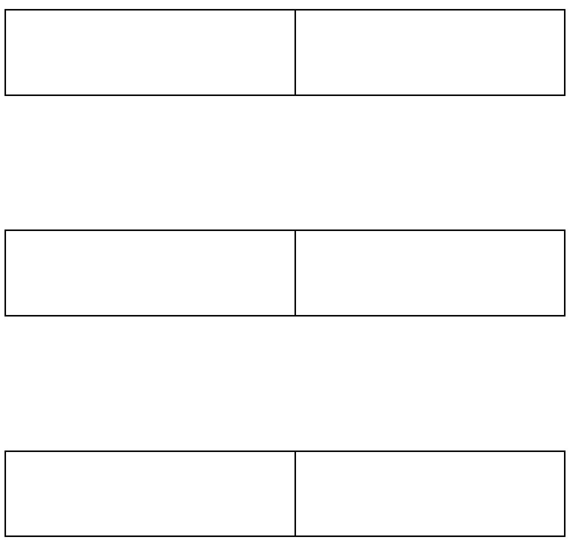
Answer Key Figure 4.2. Two pieces of construction paper equally shared by four students. Each student gets one-half of a piece of construction paper.

Motivation for this strategy: In the first part, it is natural that the unit is one piece of construction paper. In the second part, it may be easier for Kim to again select one piece of construction paper as the unit and respond that “each student gets $\frac{1}{2}$ of a piece of construction paper.”

Another possible question for Kim is:

- There are six students who share three pieces of construction paper. How can you divide the construction paper equally between the six students using the fewest number of cuts? What fraction of a piece of construction paper does each student get (Answer Key Figure 4.3)?

Answer Key Figure 4.3. Three pieces of construction paper shared by six students. Each student gets one-half of a piece of construction paper.



3. **Tom and Tiara were both asked questions about the ordering of fractions. Both of them chose to use set visual models in their responses. They each selected a specific number of objects in their sets, even though the numbers were not specified by the problem. Read Tom's response (in Figure 4.14) and Tiara's response (from Chapter 3, shown again in Figure 4.31), and answer the following questions.**

(a) Do the numbers that each selected lead to correct solutions to the problems? Explain.

Sample Answer

Tom selected 12 students for his set model. Because 12 is a multiple of 6, this model can lead him to a correct solution. Tiara selected 10 candies for her set model. Because 10 is a multiple of 10, this model can lead her to a correct solution.

(b) Are there other numbers that Tom and Tiara could have chosen? Explain.

Sample Answer

For Tom, the possible numbers of students in the class are 6, 12, 18, 24, 30, and all other multiples of 6. The answer can be found by careful consideration of the set model that Tom used. Picture the class sitting in their chairs as Tom did. To find $\frac{1}{3}$ of the class, one can make three loops that partition the class into three parts with the same number of students in each loop. This means that the number of students in the class is a multiple of 3. Similarly, we can partition the class by drawing two loops that partition the class into two parts with the same number of students in each part. This describes how Tom partitioned the class to show $\frac{1}{3}$ and $\frac{1}{2}$. Tiara's question stated that $\frac{3}{10}$ of the candies are peppermint. For someone to take out $\frac{3}{10}$ of the candies and get a whole number of candies, the number of candies must be a multiple of 10. There could be 10, 20, 30, or 40 candies, or other multiples of 10. The problem also states that $\frac{2}{5}$ of the candies are chocolate. This means that the number of candies must be a multiple of 5. But we are already confining ourselves to the multiples of 10 (which are also multiples of 5). To satisfy both conditions, the number of candies could be any multiple of 10.

(c) The solutions written by Tom and Tiara point out a unique feature of the set model as someone attempts to equipartition the set into parts that are all the same size. What is that unique feature? Describe.

Sample Answer

One can use the same linear model or the same area model to represent fractions such as $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{5}$, or $\frac{3}{10}$. However, to equipartition a set of objects so that there is the same whole number of objects in each part, the set must contain a certain number of objects. The set model can only be used when the number of objects in the set is equal to the magnitude of the denominator or is another multiple of the denominator. A set model to represent $\frac{1}{3}$ and $\frac{1}{2}$ would need to contain a number of objects that is both a multiple of 3 and a multiple of 2.

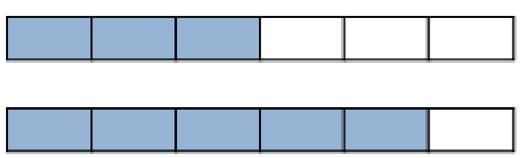
Answer Key Chapter 5: Comparing and Ordering Fractions

1. Review Ted's response in Figure 5.15. While we cannot be sure, it is possible that Ted relied on a rule to compare $\frac{5}{6}$ to $\frac{3}{6}$. What are two reasoning strategies that Ted could have used to decide who planted more corn in their garden? Describe each.

Sample Answers

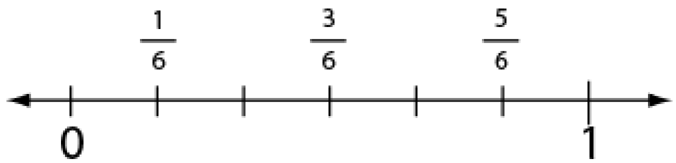
- Comparison to the reference point $\frac{1}{2}$: $\frac{5}{6}$ is greater than $\frac{1}{2}$ and $\frac{3}{6}$ is equal to $\frac{1}{2}$, therefore $\frac{5}{6}$ is greater than $\frac{3}{6}$.
- Extended unit fraction reasoning: $\frac{5}{6}$ is $\frac{1}{6}$ less than 1, and $\frac{3}{6}$ is $\frac{3}{6}$ less than 1, so $\frac{5}{6}$ is greater than $\frac{3}{6}$ because it is closer to 1.
- Models:

Answer Key Figure 5.1 Area model.



Answer Key Figure 5.2 Set Model.

Answer Key Figure 5.3 Number Line.



Mental model: Both fractions describe a whole partitioned into six equal parts, thus the sizes of the parts (sixths) described by the two fractions are equal. One fraction, however, considers five of those parts and the other fraction considers only three of those parts. Therefore, the fraction that considers more parts ($\frac{5}{6}$) is greater. It is important to note that although Ted's explanation is correct, one cannot be sure from his written response alone whether he has simply memorized a procedure or if he understands why one can compare the magnitude of the numerators when given fractions that share the same denominator.

2. Review the Everyday Mathematics Study Link in Figure 5.3 and answer the following questions.

(a) Which fraction pairs or sets of fractions provide the opportunity to use benchmarks to compare them? Explain your choices.

Sample Answers

- Problem 1: $\frac{5}{6}$ and $\frac{1}{6}$ can be compared to $\frac{1}{2}$. $\frac{5}{6} > \frac{1}{2}$ and $\frac{1}{6} < \frac{1}{2}$.
- Problem 2: $\frac{3}{10}$ and $\frac{3}{4}$ can be compared to $\frac{1}{2}$. $\frac{3}{10} < \frac{1}{2}$ and $\frac{3}{4} > \frac{1}{2}$.

- Problem 5: $\frac{4}{9}$ and $\frac{7}{9}$ can also be compared to $\frac{1}{2}$. $\frac{4}{9} < \frac{1}{2}$ and $\frac{7}{9} > \frac{1}{2}$.
- Problem 9 asks students to compare each of the fractions given to $\frac{1}{2}$.

(b) Which fraction pairs or sets of fractions provide the opportunity to use unit fraction reasoning to compare them? Explain your choices.

Sample Answer

Problems 2, 6, 10, and 11 ask students to compare fractions with common numerators. Comparing fractions with common numerators requires similar reasoning as comparing unit fractions (problem 10 is a direct unit fraction comparison). Because the numerators (the number of equal parts being considered) are the same, students must reason about how the denominator affects the number of parts the whole is divided into and thus the size of those parts.

Problems 1, 5, and 12 ask students to compare fractions with common denominators. These fraction pairs result in a whole that is partitioned into the same number of parts and the size of the parts is equal. Students must consider the number of parts (the numerator) to determine the greater fraction. This understanding can also be developed through reasoning with unit fractions.

(c) Identify fraction pairs or sets of fractions that would be difficult to compare using models. Explain your choices.

Sample Answer

Using models to make comparisons involving the fractions in problems 3, 4, 9, 10, and 11 may be difficult because, in most cases, the accuracy of partitioning decreases as the number of partitions increases, in these cases to 15ths, 40ths, 100ths, 12ths, 50ths.

(d) Problems 7 and 8 ask students to explain how they compare the fraction pairs in problems 1 and 2, respectively. Provide a instructional/assessment rationale for students explaining problems 1 and 2.

Sample Answer

Problem 7 asks students to explain how they determined the relative magnitude of $\frac{5}{6}$ and $\frac{1}{6}$. This question is important because it provides students the opportunity to generalize their understanding of comparing fractions that have the same denominator, but different numerators. Problem 8 asks students to explain how they determined the relative magnitude of $\frac{3}{10}$ and $\frac{3}{4}$. A student response to this question might provide evidence of a student's generalized understanding of comparing fractions that have the same numerator, but different denominators.

3. Read through Mark's response to the problem in Figure 5.16.

(a) Why did Mark's reasoning result in a correct solution to the problem?

Sample Answer

Mark’s response, “the more you split something the smaller the space gets,” suggests an understanding of how partitioning a whole into different numbers of pieces (i.e., 3 and 4 parts) affects the size of the parts. This reasoning results in a correct answer when comparing $\frac{1}{3}$ and $\frac{1}{4}$ because both fractions have the same numerator.

(b) Under what conditions would Mark’s reasoning not work? Explain your answer with examples.

Sample Answer

Mark’s reasoning results in a correct answer when comparing $\frac{1}{3}$ and $\frac{1}{4}$, but this would not always work with fractions that have different numerators. For example, three-fourths is greater than one-third even though thirds are greater than fourths.

(c) Provide a couple of examples of pairs of fractions you might ask Mark to compare to determine whether he can extend his unit fraction understanding to comparing other fractions. Provide a rationale for each of the fraction pairs.

Sample Answer

One might ask Mark to compare fractions such as $\frac{2}{5}$ and $\frac{2}{6}$ to help him extend his reasoning to non-unit fractions that have the same numerators. It would also be important for Mark to compare fractions in which his reasoning would not work. Comparing fractions such as $\frac{1}{4}$ and $\frac{3}{8}$ might help Mark understand that in many situations one must consider not only the size of the parts (i.e., the magnitude of the denominator) but also the number of parts (i.e., the magnitude of the numerator).

4. Read through Tom's response to the problem in Figure 5.17 and answer the following questions.

(a) What misunderstanding led Tom to conclude that both $\frac{3}{4}$ and $\frac{2}{3}$ are closest to 1?

Sample Answer

Tom concluded that $\frac{3}{4}$ and $\frac{2}{3}$ are equally close to 1 because each fraction is one part less than one whole. Three-fourths is one-fourth less than one whole, and two-thirds is one-third less than one whole. They are, however, not the same distance from the whole.

(b) What additional questions might help Tom understand why $\frac{3}{4}$ and $\frac{2}{3}$ are not the same distance from 1, even though they are both "1 away" from a whole?

Sample Answers

- Refocus the student away from the difference between the numerator and the denominator and toward the fraction away from the whole. In this case $\frac{3}{4}$ and $\frac{2}{3}$ are $\frac{1}{4}$ and $\frac{1}{3}$ away from a whole, not one part from a whole.

- Draw a model of $\frac{3}{4}$. Using the same size whole, draw a model of $\frac{2}{3}$. Do your models support your reasoning that $\frac{3}{4}$ and $\frac{2}{3}$ are equally close to one whole?

5. Read through Kim's and Bob's responses to the same problem in Figures 5.18 and 5.19.

(a) How did Kim and Bob use their knowledge of comparing proper fractions when they compared a mixed number to an improper fraction? Explain.

Sample Answer

Both Kim and Bob seemed to understand that both $1\frac{1}{2}$ and $\frac{9}{8}$ are greater than 1, and each solved the problem by determining which number is farther from 1. Both responses indicate that Susan ate more because $\frac{1}{2}$ is greater than $\frac{1}{8}$.

(b) Identify some mixed numbers/improper fractions that can be compared using benchmark reasoning to halves. Explain your choices.

Sample Answers

- $1\frac{2}{3}$ and $1\frac{1}{4}$. Both numbers are greater than 1. Since $\frac{2}{3}$ is greater than $\frac{1}{2}$ and $\frac{1}{4}$ is less than $\frac{1}{2}$, $1\frac{2}{3}$ is greater than $1\frac{1}{4}$.
- $\frac{11}{8}$ and $\frac{8}{5}$. $\frac{11}{8}=1\frac{3}{8}$ and $\frac{8}{5}=1\frac{3}{5}$. Because $\frac{3}{8}$ is less than $\frac{1}{2}$ and $\frac{3}{5}$ is greater than $\frac{1}{2}$, $\frac{11}{8}$ is less than $\frac{8}{5}$.

(c) **Identify some mixed numbers/improper fractions that can be compared using unit fraction understanding. Explain your choices.**

Sample Answers

- $\frac{5}{2}$ and $1\frac{1}{2}$. The size of the parts in both numbers is the same (i.e., halves). $\frac{5}{2} > 1\frac{1}{2}$ because $\frac{5}{2}$ has more “same size parts” than $1\frac{1}{2}$.
- $\frac{5}{3}$ and $1\frac{1}{4}$. Although each number has the same number of parts (i.e., five-thirds and five-fourths), $\frac{5}{3} > 1\frac{1}{4}$ because thirds are larger than fourths.

6. Figure 5.20 is the post-assessment response associated with the pre-assessment response shown in Figure 5.14. Identify the strategies used in Figure 5.20. Identify where each of the strategies is found on the *OGAP Fraction Progression*. Explain each of your choices.

Sample Answers

- Number 1: unit fraction. This solution compares the relative size of sixths to halves. Unit fraction reasoning is a *Fractional Strategy*.
- Solution 2: visual area models. Effectively generating a visual model to solve a problem is a *Transitional Strategy*.
- Solution 3: unit fraction and benchmarks reasoning. In this solution, the unit fraction reasoning stated in solution 1 is applied to conclude that $\frac{1}{2} > \frac{1}{6}$. Benchmark reasoning is

used to conclude that $\frac{7}{9}$ and $\frac{11}{13}$ are “greater than $\frac{1}{2}$.” Both unit fraction and benchmark reasoning are *Fractional Strategies.*

- Solution 4: unit fraction reasoning. This solution references the distance from the whole for each fraction ($\frac{2}{13}$ and $\frac{2}{9}$) and the relative size of the parts (“13ths are smaller”) to conclude that $\frac{11}{13}$ is closer to 1 than $\frac{7}{9}$. Again, unit fraction reasoning is a *Fractional Strategy*.

Answer Key Chapter 6: Number Lines and Fractions

1. To help explore the relationship between measurement and number lines, respond to the following questions.

(a) What are at least three important properties that number lines and measurement tools (such as rulers) share that have the potential to facilitate students' understanding of the connections between rulers and number lines?

Sample Answers

Both number lines and rulers have these same basic properties:

- The numbers on a ruler are positioned proportionally. It is the same distance from 0 to 1, from 1 to 2, from 2 to 3, and so forth on a ruler, and it is the same distance from 0 to 1, from 1 to 2, from 2 to 3, and so forth on a number line.
- The units are continuous.
- Units can be subdivided.
- They measure a distance from zero.

(b) What are two important differences between number lines and scales on measuring tools?

Sample Answers

- An important difference between number lines and scales on measuring tools is that scales on measuring tools have standard units (same length given the defined unit—e.g., inch) no matter how many units are on the scale. On the other hand, the size of the unit can vary from number line to number line.
- Units on number lines can be divided into any number of subdivisions, whereas the subdivisions on rulers and other measuring tools are usually predefined and consistent with systems of measurement (e.g., metric: denominators are powers of 10, English: denominators are powers of 2).

(c) You provide your students with inch rulers, centimeter rulers, and strips of paper to measure. Their task is to measure each of the strips to the nearest eighth of an inch and tenth of a centimeter. Before your students begin using the rulers to measure the strips, identify three similarities and three differences between an inch ruler and a centimeter ruler. Use Figure 6.27.

Sample Answers

- On the centimeter ruler in Figure 6.27, the zero is not located at the end like it is on the inch ruler.
- The tick marks on the inch ruler use different lengths to indicate inches, half inches, quarter inches, and eighth inches. Those markings can help the eye differentiate between various parts of an inch. The decimal parts of a centimeter have tick marks that are all same length except for the longer mark used to show half centimeters.

- Since one-tenth of a centimeter is smaller than one-eighth of an inch, more accurate measurements can be made with the centimeter ruler than with the inch ruler.

2. Mr. Brown had a large number line in the front of his classroom (Figure 6.28). On the first day that he used the number line, he asked some students to place $\frac{1}{2}$ on the number where they thought it belonged. Mr. Brown had done no prior instruction with number lines, but he thought this would be a good way to get information about what instructional issues he might face as students begin using number lines to solve fraction problems.

(a) The students were unsure where to locate $\frac{1}{2}$, but decided to locate it at the 3 on the number line. Is this correct or incorrect? Explain.

Sample Answer

Their answer is incorrect. They found $\frac{1}{2}$ of the length of the line, not the location of $\frac{1}{2}$ on the number line.

(b) What feature(s) of the number line may have been ignored by these students?

Sample Answer

The symbols that define the units have been ignored.

3. **Look at Matt's response to the problem in Figure 6.29. What understandings/misunderstandings are evidenced in Matt's response? Describe the evidence.**

Sample Answer

It is not clear which partitions on the number line Matt did first. However, when he partitioned the number line into twelfths, he disregarded the sixths that were provided on the number line. Therefore, relative to the sixths, twelfths were not accurately placed on the number line. It also appears that the fourths were placed to fit between some twelfths, disregarding both the sixths and the twelfths.

Based on the evidence in this response, Matt used little understanding of equivalence to locate the fractions on the number line (e.g., $\frac{6}{12} = \frac{3}{6}$ or $\frac{9}{12} = \frac{3}{4}$).

4. **Nick (Figure 6.30) placed $\frac{1}{3}$ below 0 on the number line.**

(a) **What reasoning did Nick use to solve the problem? Describe the evidence.**

Sample Answer

Nick does not consider $\frac{1}{3}$ to be a positive number, so he placed $\frac{1}{3}$ below 0.

(b) **What are some potential next instructional steps for Nick given the evidence in his work?**

Sample Answer

Nick applies the misconception that because $\frac{1}{3}$ is not a whole number, it must be less than zero. It is possible that Nick, when asked, could partition an area model into thirds and shade $\frac{1}{3}$ of it. To help Nick unravel this misconception, one might use concrete materials, such as a candy bar, and ask: Show me $\frac{1}{3}$ on this candy bar. Once he shows you a third of a candy bar, ask: Is this more or less than a whole candy bar? Is it more or less than 0? Assuming that he realizes that $\frac{1}{3}$ is more than 0 and less than 1, ask him to reconsider the location of $\frac{1}{3}$ on the number line.

5. ***OGAP Fraction Progression:*** **Review student work in Figures 6.9, 6.10, 6.11, 6.13, 6.15, and 6.24 in this chapter using the** ***OGAP Fraction Progression.*** **For each piece of student work:**

(a) Use the evidence in the student work to locate the work on the framework.

(b) Provide a rationale for your decision.

(c) Discuss some possible next instructional steps or feedback you might give to the student based on the evidence in the student work.

Sample Answers

Student work	**Location on *Fraction Progression***	**Rationale**	**Possible next instructional steps**
Figure 6.9	*Early Fractional Strategy*	Sequential rather than proportional reasoning	Provide opportunities to partition area and linear models into equal parts. Perhaps begin by having the student locate the whole numbers on a number line.
Figure 6.10	*Transitional Strategy*	Used an area model to help locate fractions on a number line	Provide opportunities to use equipartitioning to locate fractions on a number line without the use of an additional model. Perhaps begin with whole numbers and unit fractions.
Figure 6.11	*Fractional Strategy*	Accurately locates fractions on a number line	Increase the complexity of the fractions to include negative fractions and fractions and fractions and mixed numbers greater than 1.

Continued

Continued

Figure 6.13	*Fractional Strategy*	Accurately locates fractions on a number line	Increase the complexity of the fractions to include non-unit fractions and fractions and mixed numbers greater than 1. Also alter the number line to include a larger span and different partitions.
Figure 6.15	*Fractional Strategy*	Accurately locates fractions on a number line	Increase the complexity of the fractions to include negative fractions and mixed numbers greater than 1.
Figure 6.17	*Transitional Strategy*	Used an area model to help locate fractions on a number line	Provide opportunities to use equipartitioning to locate fractions on a number line without the use of an additional model. Perhaps begin with whole numbers and unit fractions.
Figure 6.24	*Transitional Strategy*	Generates a visual model to solve a problem	Provide opportunities for the student to reason about the magnitude of fractions and order fractions according to their magnitude. One would want to move the student from reliance on a model to strategies such as unit fraction reasoning, reasoning about relative magnitude, and benchmark reasoning.

Answer Key Chapter 7: The Density of Fractions

1. Review Seth's response in Figure 7.19 and then answer the following questions.

(a) How did Seth use his understanding of partitioning to answer part A of the question?

Sample Answer

Seth partitioned an area model. He made models the same size and partitioned accurately. This strategy allowed him to identify other fractions located between $\frac{1}{3}$ and $\frac{3}{4}$.

(b) Based on Seth's response to part B, what are the strengths and limitations of his partitioning strategies?

Sample Answer

The strength of using models is that one can visualize the space for locating other fractions. This strategy is ultimately limiting based upon the size of the model, as was evidenced in Seth's response to part B. Although he was not sure that there were other fractions between $\frac{1}{3}$ and $\frac{3}{4}$, he did identify one, $\frac{5}{8}$. Seth did not conceptualize that there are infinite fractions between these two fractions.

2. It was suggested in the chapter that number lines could be used to extend the developing understandings of Richard (Figure 7.11), Madison (Figure 7.12 and

Todd (Figures 7.13 and 7.16). Review each response and then answer the following questions.

(a) Richard (Figure 7.11) named only one fraction. Provide an example of a way the number line could extend his understanding to identifying different fractions between $\frac{1}{3}$ and $\frac{3}{4}$ besides $\frac{1}{2}$.

Sample Answer

Richard named $\frac{1}{2}$ with no transparent justification. One might hypothesize that he answered $\frac{1}{2}$ based on a benchmark understanding that $\frac{1}{3} < \frac{1}{2}$ and $\frac{3}{4} > \frac{1}{2}$. First, ask Richard to explain his reasoning. To use a number line to extend Richard's understanding, one might provide a number line labeled with only $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{3}{4}$ and say, "This is a number line with $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{3}{4}$ on it. You are right that $\frac{1}{2}$ is between $\frac{1}{3}$ and $\frac{3}{4}$, but there is also more space between these fractions. Can you name another fraction that might be in this space? Explain why or why not." If Richard is still confused, take out a ruler that is already partitioned to eighths and say, "Let's look at this ruler. Show me where $\frac{1}{4}$ and $\frac{3}{4}$ are found on the ruler. What fractions can you name between these fractions?"

(b) Madison named equivalent fractions instead of other fractions. Provide an example of a way the number line could extend her thinking beyond equivalent fractions.

Sample Answer

Madison named equivalent fractions. To use a number line to extend her understanding, one might provide a number line labeled with the fractions she named and say, "This is a number line with the fractions you named on it. You are right that $\frac{2}{3}$ is between $\frac{1}{3}$ and $\frac{3}{4}$, but notice that $\frac{4}{6}$ is at the same location on the number line. This shows that $\frac{2}{3}$ and $\frac{4}{6}$ are equivalent fractions [see Chapter 8 for a discussion of different names for the same number]. Can you name another fraction, not equivalent to $\frac{2}{3}$, that is between $\frac{1}{3}$ and $\frac{3}{4}$? Explain why or why not." Focus the student's attention on the empty space on the number line. It might be helpful to use a ruler as discussed in the answer to the previous question.

(c) Todd (Figures 7.13 and 7.16) used a number line to identify some fractions, but did not extend that to a more general understanding. Provide an example of a way the number line could extend his thinking beyond equivalent fractions.

Sample Answer

Unlike Richard and Madison, Todd's solution is based on a number line. To move Todd to a new level of understanding, focus him on the space between each of the numbers that he named by saying, "Todd, you used your number line to identify some fractions between $\frac{1}{3}$ and $\frac{3}{4}$ by partitioning your number line into twelfths. You found five fractions between $\frac{1}{3}$ and $\frac{3}{4}$. Let's look at $\frac{7}{12}$ and $\frac{8}{12}$, two of the fractions you named. I notice that there is space on the number line between these two numbers. Do you think there is another number located between $\frac{7}{12}$ and $\frac{8}{12}$?

Why or why not?" If Todd repartitions the line to show $\frac{15}{24}$, ask, "Are there any numbers between $\frac{15}{24}$ and $\frac{7}{12}$?" Continue this line of questioning.

3. Find three different fractions between the following fraction pairs using two different strategies for each fraction pair. Then answer the questions that follow.

$\frac{4}{10}$ **and** $\frac{7}{10}$

$\frac{1}{8}$ **and** $\frac{1}{4}$

$\frac{1}{10}$ **and** $\frac{1}{9}$

Sample Answer

Some common strategies that can be used with different fraction pairs include: modeling, common denominators, common numerators, conversion to decimals or percents, unit fraction reasoning, use of complex fractions (i.e., fractions with fractions in the numerator or in the denominator).

Some fractions between $\frac{4}{10}$ and $\frac{7}{10}$:

- $\frac{5}{10}$, $\frac{6}{10}$, $\frac{11}{20}$ are between $\frac{4}{10}$ and $\frac{7}{10}$ (common denominators, modeling)

Some fractions between $\frac{1}{8}$ and $\frac{1}{4}$:

- $\frac{1\frac{1}{2}}{8}$, $\frac{5}{16}$, $\frac{6}{16}$ are between $\frac{1}{8}$ and $\frac{1}{4}$ (common denominators or modeling)
- 0.13, 0.14, and 0.15 are between 0.125 ($\frac{1}{8}$) and 0.25 ($\frac{1}{4}$) (conversion to decimal equivalents)
- $\frac{3}{23}$, $\frac{3}{22}$, $\frac{3}{21}$, . . ., $\frac{3}{13}$ (common numerator, $\frac{3}{24} = \frac{1}{8}$ and $\frac{3}{12} = \frac{1}{4}$)

Some fractions between $\frac{1}{10}$ and $\frac{1}{9}$:

- $\frac{4}{39}$, $\frac{4}{38}$, $\frac{4}{37}$ (common numerator of 4 [$\frac{4}{40}$ and $\frac{4}{36}$])
- $\frac{1}{9.5}$ (fractional denominator with unit fraction reasoning; because 10ths are smaller pieces than 9ths, a fraction with a denominator of 9.5 would be between the two fractions $\frac{1}{10}$ and $\frac{1}{9}$)

(a) What difficulties do you think students might encounter as they solve these problems?

Sample Answer

Unless students have had experience with a range of reasoning strategies that includes modeling, equivalent forms of fractions (decimals, percents), common numerators, common denominators, unit fractions reasoning, and exposure to fractional numerators and denominators, they may be limited in their strategies.

(b) What kinds of errors might result from these difficulties?

Sample Answers

- Students may revert to providing equivalent fractions instead of other fractions.
- Models may result in incorrect solutions.
- Use of reasoning with fractional numerators and denominators may result in identifying the incorrect fractions.

- Students may convert the fractions to decimals and percents instead of using fractional reasoning.

(c) As a set of questions, what information can the student provide that the evidence from a single question might not provide?

Sample Answer

Because the fraction pairs get progressively closer together, the need for flexibility increases. For example, it is possible to effectively use models for the first two pairs. However, they are not very effective when identifying fractions between the last pair. As a set, they can provide evidence of the student's flexibility and understanding of the density of rational numbers.

Answer Key Chapter 8: Equivalent Fractions and Comparisons

1. What is the evidence in Emma's response in Figure 8.15 that demonstrates understanding equivalence in this situation?

Sample Answer

Emma showed an understanding of equivalence in three ways: (1) her model showed that $\frac{4}{8}$ is equivalent to $\frac{1}{2}$; (2) she provided the explanation "because $\frac{1}{2}$ is equal to $\frac{4}{8}$"; and (3) she wrote the equation, $\frac{4}{8}+\frac{3}{8}=\frac{7}{8}$.

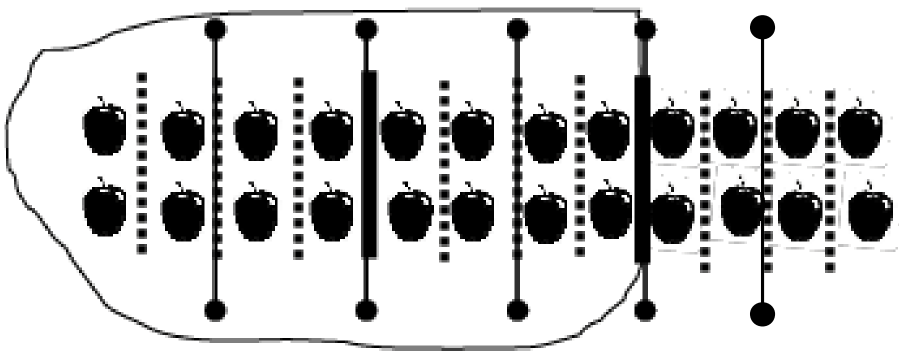
2. Use models to address the following:

(a) Illustrate that $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{8}{12}$ are equivalent using area models, set models, or number lines.

Sample Answer

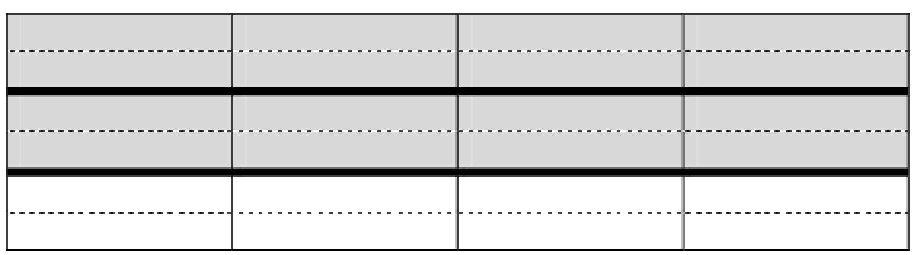
Using a set of 24 apples (Answer Key Figure 8.1), one can partition the set into 12 equal groups (dotted partitions), 3 equal groups (heavy line partitions), and 4 equal groups (fine line partitions). In this way, one can see that $\frac{2}{3}$ of the set of 24 apples, $\frac{4}{6}$ of the set of 24 apples, and $\frac{8}{12}$ of the set of 24 apples are the same number of apples. Therefore, $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{8}{12}$ are equivalent fractions.

Answer Key Figure 8.1. A set of 24 apples partitioned into thirds, sixths, and twelfths modeling the equivalence between $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{8}{12}$.



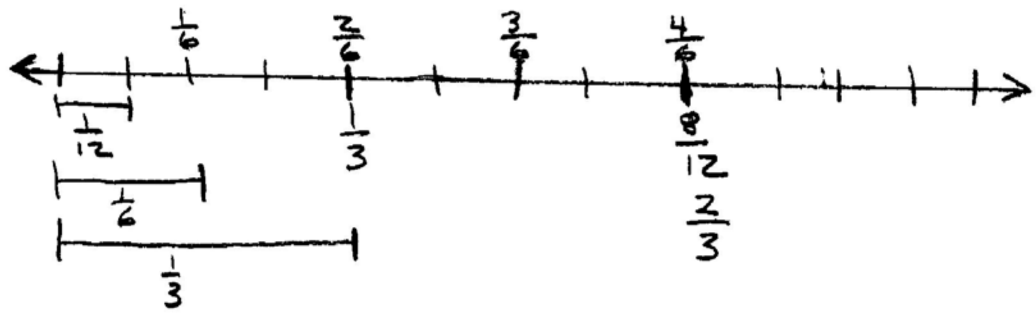
The area model in Answer Key Figure 8.2 is partitioned into thirds (heavy partitions), sixths (half of each third), and twelfths (two small rectangles). The shaded area represents $\frac{4}{6}$ of the figure (four of the six rows are shaded), $\frac{2}{3}$ of the figure**,** and $\frac{8}{12}$ of the figure, illustrating that $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{8}{12}$ are equivalent.

Answer Key Figure 8.2. An area model showing the equivalence between $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{8}{12}$.



The number line in Answer Key Figure 8.3 is partitioned into twelfths and then repartitioned into thirds and sixths. The number line is a powerful tool to show that $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{8}{12}$ are equivalent because they are all located at the same place on the number line.

Answer Key Figure 8.3. Number line.

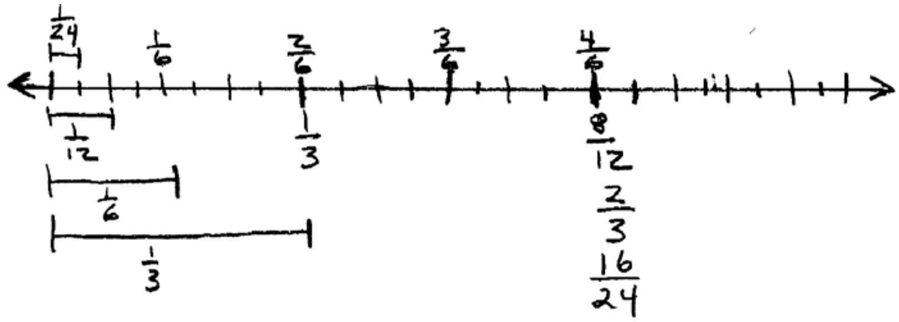


(b) Name one more fraction that is equivalent to $\frac{2}{3}$. Adapt one of your models in part a to show that the fraction is equivalent to $\frac{2}{3}$.

Sample Answer

The number line from Answer Key Figure 8.3 is repartitioned into 24ths (Answer Key Figure 8.4), showing that $\frac{16}{24}$ is equivalent to $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{8}{12}$. To find another equivalent fraction, one can repartition the 24ths into 48ths and find that $\frac{32}{48}$ is equivalent to $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{8}{12}$.

Answer Key Figure 8.4. Number line.



3. Review Kenny's response (Figure 8.16) and then answer the questions that follow.

(a) What is the evidence in Kenny's response that demonstrates understanding equivalence in this situation?

Sample Answer

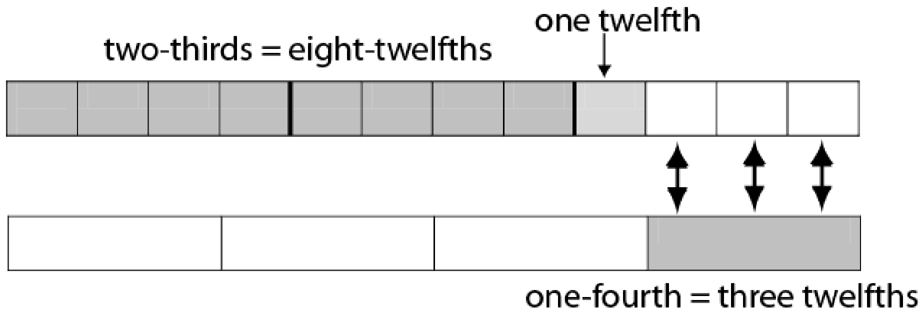
Kenny found equivalent fractions $\frac{2}{3}=\frac{8}{12}$ and $\frac{1}{4}=\frac{3}{12}$. This allowed him to calculate eight-twelfths + three-twelfths = eleven-twelfths. In addition, Kenny used his knowledge that $\frac{12}{12}$ is equivalent to 1. This allowed him to subtract eleven-twelfths from twelve-twelfths.

(b) How might a student select a model to use in solving this problem? Show how the model you select can help build an understanding of equivalence.

Sample Answer

The area models in Answer Key Figure 8.5 illustrate the difference (one-twelfth) between the sum of eight-twelfths plus three-twelfths and the whole (twelve-twelfths). The model also shows that $\frac{2}{3}=\frac{8}{12}$ and $\frac{1}{4}=\frac{3}{12}$.

Answer Key Figure 8.5. Area models.



4. Chris accurately calculated the distance in the problem in Figure 8.17.

(a) What is the evidence in Chris's response that he understands equivalence?

Sample Answer

It is unclear whether Chris is using a procedure with understanding. However, Chris was able to find equivalent fractions. He converted the mixed numbers to equivalent improper fractions and he found equivalent fractions for 7 thirds (35 fifteenths) and 9 fifths (27 fifteenths). This allowed him to accurately solve the problem.

(b) What concerns do you have about his solution?

Sample Answer

Chris appears to use a procedure that involves multiplying the numerator of one fraction by the denominator of another fraction and then multiplying the denominators by each other to find a common denominator. Although one cannot be sure without asking, it appears that the procedure is not linked to understanding of equivalence. One wonders how Chris would apply this strategy of finding equivalent fractions if there were three or more fractions to be added. The strategy appears to have a limited application.

Answer Key Chapter 9: Addition and Subtraction of Fractions

1. Mrs. Grayson brought Kenny's work (shown first in Figure 9.1 and shown here in Figure 9.25) to a fifth-grade team meeting. She wondered if Kenny simply followed a procedure or if he understood the concepts upon which the algorithm is based. Help Mrs. Grayson by answering the following:

a) Describe evidence in Kenny's response that shows understanding of the context of the problem and related fraction concepts.

Sample Answer

Kenny's appropriate use of both addition ($\frac{2}{3}+\frac{1}{4}=\frac{11}{12}$) and subtraction ($1-\frac{11}{12}=\frac{1}{12}$) provides evidence that he understands the context of the problem. In addition, the evidence in his work shows an ability to find and use equivalent fractions (e.g., $\frac{2}{3}=\frac{8}{12}$, $\frac{1}{4}=\frac{3}{12}$, and $\frac{12}{12}=1$).

b) What questions might you ask Mrs. Grayson about her instruction to ensure that Kenny has a foundation for understanding?

Sample Answers

- Have students used models not only to solve fraction problems involving equivalence, addition, and subtraction, but also to generate mathematical understanding of these concepts?

- Has your instruction provided plenty of opportunities for students to determine the correct operation(s) needed to solve a variety of fraction problems?
- Have your students had numerous experiences estimating fraction sums and differences?

(c) If Mrs. Grayson wanted to be sure that Kenny understood the algorithm, what else could she ask him?

Sample Answers

To be sure Kenny understood the algorithm, Mrs. Grayson might ask him to:

- Draw a model to illustrate the calculations he made.
- Estimate the answer to a similar two-step problem and explain how he arrived at this estimated answer.
- Solve a similar two-step problem with more complex fractions (e.g., use fifths, sixths, mixed numbers). Ask him to estimate and calculate the exact answer.

2. Mr. Benson brought Mathew's response (Figure 9.26) to the team meeting. He felt that this provides evidence that Mathew has a strong conceptualization when comparing $\frac{2}{5}$ to $\frac{3}{10}$ using both a visual area model and a number line. Answer the following questions:

(a) What understandings are evidenced in Mathew's work?

Sample Answer

Mathew appears to understand how to use an area model and a number line to compare the fractions $\frac{2}{5}$ to $\frac{3}{10}$. He was able to partition his models with enough accuracy to determine that $\frac{2}{5} < \frac{3}{10}$.

(b) What questions could be asked to build understanding about equivalence and common denominators when comparing or adding and subtracting fractions? Explain how each question might help Mathew move to a deeper understanding of equivalence and common denominators when comparing or adding and subtracting fractions.

Sample Answers

How does the "size" of $\frac{1}{10}$ compare to the "size" of $\frac{1}{5}$? Use either your area or linear models to support your answer.

Using either your area or linear models for reference, determine how many 10ths are equivalent to $\frac{1}{5}$.

Use your answer to the first question to:

a. Figure out how many tenths are equivalent to $\frac{2}{5}$?

b. Determine the sum of $\frac{2}{5} + \frac{3}{10}$.

c. Determine the difference of $\frac{2}{5} - \frac{3}{10}$.

d. Use your linear models to check your answers to questions a–c.

The first two questions ask Mathew to consider the relationship between tenths and fifths; it takes two one-tenths to equal one-fifth or an understanding that tenths are one-half the “size” of fifths. The remaining questions help Mathew use this understanding to answer equivalence, addition, and subtraction problems that can be checked using his models.

3. Ms. Cunningham shared Kim’s work (Figure 9.27) at the math team meeting. She is asking her teammates for advice on how to transition students to accurately using models to solve problems involving addition and subtraction. Help Ms. Cunningham by addressing the following questions:

(a) What did Kim visually model correctly? What is the evidence?

Sample Answer

Kim drew circle models to represent $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{3}{7}$.

(b) Kim’s model leads to an incorrect response. What errors did Kim make in her modeling? What is the evidence?

Sample Answer

Kim appeared to add the total number of shaded pieces in her models for $\frac{2}{3}$ and $\frac{1}{4}$ to determine the numerator in her answer. To establish the denominator in her sum, Kim added the total number of pieces in her models for $\frac{2}{3}$ and $\frac{1}{4}$. Kim seemed to consider only the number of pieces with no regard for the size of the pieces.

(c) What questions might you ask, or activities might you do, to help Kim understand how to use models to solve addition and subtraction problems?

Sample Answers

- Ask Kim to first use her models to estimate the sum of $\frac{2}{3}+\frac{1}{4}$. If the shaded pieces of her models for $\frac{2}{3}$ and $\frac{1}{4}$ were combined, how would the new shaded portion compare to $\frac{1}{2}$ of the circle? To one whole circle?
- One might encourage Kim to use rectangular area models. They may be easier to partition and it may be easier to interpret the results of partitioning.
- It would be helpful for Kim to gain experience placing fractions on a number line. This would emphasize the concept that fractions are numbers and help her gain a stronger sense of the magnitude of fractions. A stronger understanding of magnitude would help her see that the sum of $\frac{2}{3}+\frac{1}{4}$ (a fraction greater than $\frac{1}{2}$ added to a fraction less than $\frac{1}{2}$) could not be $\frac{3}{7}$ (less than $\frac{1}{2}$).

4. Mr. Hill has been spending a lot of time working with his class on estimating sums and differences. Work from three of this students, Willy, Oscar, and Christine, is

shown in Figures 9.28, 9.29, and 9.30. Use these three solutions to answer the following questions.

(a) Analyze Willy's, Oscar's, and Christine's work. What strategy did each student use to solve the problem? Locate each solution strategy on the *OGAP Fraction Framework.*

Sample Answers

- Willy reasoned about the relative magnitude of both $\frac{7}{8}$ and $\frac{1}{12}$, then compared the sum to the numbers provided in the problem. This is a *Fractional Strategy* and is efficient, given the problem.
- Oscar seemed used visual models to represent both $\frac{7}{8}$ and $\frac{1}{12}$ accurately enough for the problem given. He also realized that the shaded portion representing $\frac{7}{8}$ together with the shaded portion representing $\frac{1}{12}$ would be close to one whole. This is a *Transitional Strategy*.
- Christine appears to understand how to use common denominators to add $\frac{7}{8}+\frac{1}{12}$. She also correctly interpreted the meaning of her sum, $\frac{92}{96}$. That is, she seemed to understand that $\frac{92}{96}\approx 1$. This is a *Fractional Strategy*, although it is not the most efficient strategy for this particular problem.

(b) What questions might you ask Christine to help her consider more efficient fractional strategies?

Sample Answers

Suggested activities or questions for Christine:

- Christine might benefit from opportunities to differentiate between problem situations that require an exact answer and problems for which an estimated answer is needed. In this case, one would want Christine to see that this problem called for an estimate.
- It might also be helpful for Christine to model fraction addition problems to help her reason about the relative magnitude of fractions. Although there is evidence in her solution that Christine understood the magnitude of $\frac{92}{96}$ in relation to 1, she did not use a strategy that called for an understanding of the magnitude, as Oscar did in Figure 9.19.

(c) What questions or activities could you propose that would help Oscar move from using a visual model to a mental visual model of the fractions being added? How would the questions or activities you propose help Oscar?

Sample Answers

- Ask Oscar how $\frac{7}{8}$ and $\frac{1}{12}$ compare to benchmarks such as 1, $\frac{3}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$. Ask Oscar to use his model to support his answers.
- Ask Oscar to describe the relative "size" of eighths. Ask him to use this understanding to reason about the relative size of $\frac{7}{8}$.
- Ask Oscar to describe how the size of eighths compares to the size of twelfths. These questions are intended to help Oscar build upon his understanding of modeling and develop a more abstract understanding when estimating fraction sums.

(d) **Use the *OGAP Fraction Framework* to help you create other questions or activities that can support or extend Willy's understanding of the relative magnitude of** $\frac{1}{12}$ and $\frac{7}{8}$.

Sample Answer

One would want to alter the types of problems for Willy to extend this reasoning to other types of fractions, such as fractions greater than one, mixed numbers, and negative fractions. Estimation problems involving subtraction might also be valuable for Willy. One might also consider problems similar to $\frac{1}{12}+\frac{7}{8}$ that are in a context.

5. Ms. Horton is helping Emmanuel (Figure 9.31) understand addition of proper fractions. He is able to draw visual models for most fractions and use visual models to add or subtract fractions with common denominators. However, Emmanuel struggles with adding or subtracting fractions with unlike denominators (Figure 9.32). Help Ms. Horton by responding to the following questions and prompts.

(a) **What feature of the model in Figure 9.31 allowed Emmanuel to successfully add the two fractions but is not present in Figure 9.23? Locate each solution strategy on the *OGAP Fraction Framework.***

Sample Answer

In Figure 9.31, Emmanuel's models for $\frac{1}{8}$ are $\frac{3}{8}$ are equipartitioned into same-sized pieces. This means that the numerator of his sum, $\frac{4}{8}$, can be determined by counting the total number of shaded pieces in his models. Because the fractions given in the problem share the denominator 8, there was no need for Emmanuel to repartition either model in order to add the fractions. The solution in Figure 9.31 is a *Transitional Strategy*. He used a visual model to solve the problem. Emmanuel's solution in Figure 9.32 is an *Early Fractional Strategy*. Emmanuel used visual models appropriate for the situation, but the solution contains errors.

(b) What would Emmanuel have to do to the model in Figure 9.32 to allow him to effectively use the same strategy for adding fractions as he used in Figure 9.31?

Sample Answer

Emmanuel drew models that accurately represent both $\frac{1}{2}$ and $\frac{3}{8}$. Because the denominators given in the problem define different-sized pieces, Emmanuel would have to repartition the models so that the pieces in each model are the same size. The most efficient way to do this would be to repartition his model of one-half into eighths.

(c) Provide a sequence of addition/subtraction problems that would help build this understanding. Describe how the problems you propose can help to build an understanding of the meaning of common denominators.

Sample Answers

- Use models to determine the sum of $\frac{1}{2}+\frac{1}{4}$. This fraction pair allows Emmanuel to use a halving strategy to repartition. It might be helpful for Emmanuel to estimate the sum prior to creating the models. It is possible that Emmanuel already knows the answer to this question. In that case, we are helping Emmanuel consider repartitioning and ultimately common denominators by building on preexisting knowledge.

- Use models to determine the sum of $\frac{1}{2}+\frac{1}{8}$. This fraction pair also provides Emmanuel the opportunity to use a halving strategy. Unlike the first pair suggested, Emmanuel needs to partition in two steps: halves into fourths, fourths into eighths. Again, estimating prior to modeling can be helpful. It is important to help Emmanuel see that when the two models are partitioned into equal-sized pieces, they are similar to the models in Figure 9.32 that he constructed to successfully solve the first question. Again, one is helping Emmanuel build on prior understandings.
- Use models to determine the sum of $\frac{1}{2}+\frac{1}{3}$. Since one cannot use a halving strategy to partition halves into thirds, this pair may present new challenges for Emmanuel. Emmanuel might require considerable opportunities to solve problems such as these (e.g., $\frac{1}{3}+\frac{1}{4}$, $\frac{2}{3}+\frac{1}{4}$, $\frac{1}{2}+\frac{2}{5}$). Once Emmanuel understands how to model a variety of fraction pairs, one might consider fractions whose sums are greater than 1, such as $\frac{3}{4}+\frac{1}{2}$. It is important to remember that, ultimately, Emmanuel needs to move to a more abstract understanding of addition of fractions and develop procedural fluency. Keep in mind that the use of models, like we are describing here, is important in the development of procedural fluency but is not the ultimate goal.

Answer Key Chapter 10: Multiplication and Division of Fractions

1. **Mr. Way gave his class a pre-assessment prior to the upcoming unit on multiplication and division of fractions. He is concerned about Claudia's response to the division problem in Figure 10.27. Help Mr. Way by answering the following questions.**

(a) **What are some possible explanations for Claudia's apparent belief that "the division sign means to multiply"?**

Sample Answer

It is likely that Claudia is applying the multiplication step of the algorithm (invert and multiply) without understanding when or why she should multiply. In addition, fraction multiplication or division problems can often be solved using either operation. For example, to determine $\frac{1}{4}$ of 12 books, one could calculate 12 ÷ 4 = 3 or $12 \times \frac{1}{4} = 3$. This can be confusing for some students and lead them to believe, as Claudia does, that the "division sign means to multiply."

(b) **What are some questions, lessons, or activities that Mr. Way could use to help Claudia develop an understanding of the similarities and differences between multiplication of fractions and division of fractions?**

Sample Answers

- Provide Claudia plenty of opportunities to model multiplication of fraction and division of fraction problems and to interpret the models that she constructed. One goal of this emphasis on models should be to help Claudia see division as “how much or how many” of one quantity there is in the other.
- When the divisor is smaller than the dividend, a student’s understanding can be helped by asking “how many” (e.g., $1\frac{1}{2} \div \frac{1}{4}$ can be interpreted as how many $\frac{1}{4}$s are there in $1\frac{1}{2}$).
- When the divisor is greater than the dividend, a student’s understanding can be helped by asking “how much” (e.g., $\frac{1}{4} \div \frac{1}{2}$ can be interpreted as how much of $\frac{1}{2}$ is in $\frac{1}{4}$). Claudia must be able to differentiate this division interpretation from multiplication. Claudia could interpret multiplication as a “fractional part of a fractional part.” For instance, $\frac{1}{4} \times \frac{3}{4}$ can be thought of as one-fourth of a piece whose size is three-fourths.
- Provide many contexts in which Claudia must determine the operation (multiplication or division) required to solve the problem.
- Give Claudia opportunities to estimate multiplication of fraction and division of fraction problems.

2. The strategy shown in Figure 10.22 is representative of Corey’s solutions for fraction division problems. Corey’s teacher, Mrs. Rousseau, would like Corey to use her understanding of models to develop a more efficient algorithmic approach for division of fraction problems. Examine Corey’s response in Figure 10.22 and answer the questions that follow.

(a) Based on the evidence, what concepts related to division of fractions does Corey appear to understand?

Sample Answer

Corey appears to understand that this problem can be interpreted as: How many $\frac{3}{4}$s are there in 6? She modeled this situation with a number line that allowed her to count the number of $\frac{3}{4}$s until she reached 6 pounds.

(b) How could Mrs. Rousseau use the developing understanding you identified in part a and her facility with models to help Corey develop an algorithmic approach for solving division of fractions problems?

Sample Answer

Mrs. Rousseau can help Corey interpret her model in way that facilitates a common denominator strategy for dividing by a fraction.

- Use the model to understand that the 6 pounds of candy can be interpreted as $\frac{24}{4}$ pounds of candy.
- This context can be seen as $\frac{24}{4} \div \frac{3}{4}$, or how many $\frac{3}{4}$s are there in $\frac{24}{4}$. The number of $\frac{3}{4}$s in $\frac{24}{4}$s (24 fourths ÷ 3 fourths) is no different from calculating the number of groups of 3 balls in 24 balls (24 balls ÷ 3 balls). So $\frac{24}{4} \div \frac{3}{4} = 8$, just like 24 balls ÷ 3 balls = 8.

Note: There are certainly other procedures that Mrs. Rousseau could help Corey understand. We chose the common denominator strategy in this case because Corey's number line perfectly models this procedure.

3. Despite the fact that Ms. Altrui's class can use models effectively to solve equivalence, magnitude, addition, and subtraction problems, the group is struggling with using models to solve more complex multiplication and division of fraction problems. Help Ms. Altrui by studying Tracy's solution in Figure 10.24 and answering the questions that follow.

(a) What context does Tracy's model appear to represent?

Sample Answer

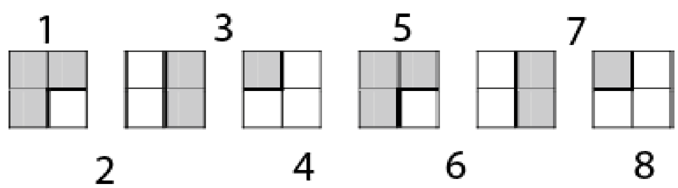
Tracy's model shows 6 bags of candy, each bag three-fourths full.

(b) How could Tracy's model be modified or reinterpreted to answer the question posed in the problem?

Sample Answer

Tracy's area model is partitioned into fourths, similar to Corey's number line in question 2. This is an important feature of Tracy's model and it can be used to help answer the question. In order to count the number of $\frac{3}{4}$ bags needed, Corey would have to reinterpret her model similar to the example shown in Answer Key Figure 10.1. Notice how a slight modification to Tracy's model can help provide clarity for this division context.

Answer Key Figure 10.1. Tracy's modified area model showing that there are 8 ($\frac{3}{4}$) in 6.



(c) Identify questions, activities, or lessons that Ms. Altrui could use to help her class extend their models of equivalence, magnitude, and addition and subtraction problems to include effective models for multiplication and division.

Sample Answer

Ask students to use fraction models to answer questions such as:

- How many $\frac{1}{8}$s are there in $\frac{1}{2}$?
- How much of $\frac{1}{2}$ is in $\frac{1}{8}$?
- Show $\frac{1}{4}$ of $\frac{1}{2}$.
- Show $\frac{2}{3}$ of $\frac{3}{4}$.
- How many times greater is $\frac{1}{2}$ than $\frac{1}{4}$?
- What would you have to divide $\frac{3}{4}$ by to equal $\frac{1}{8}$?

- What would you have to multiply $\frac{2}{3}$ by to equal 4?
- Show how multiplying a number by $\frac{1}{2}$ gives the same answer as dividing the number by 2.

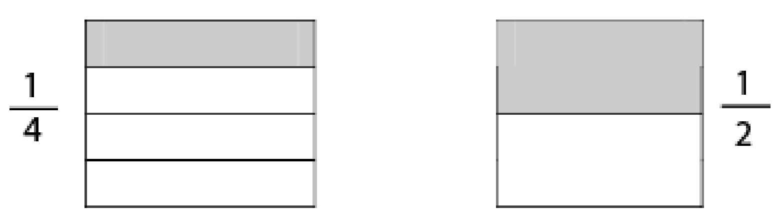
4. Although Alejandro can solve both multiplication and division of fraction problems, he tends to confuse the two operations. He often misinterprets problems requiring a division strategy and solves them using multiplication. Figure 10.26 is an example of his confusion.

(a) How could Alejandro's model be altered or reinterpreted to answer the question $\frac{1}{2} \div \frac{1}{4}$ = ?

Sample Answer

Alejandro's solution accurately models $\frac{1}{2} \times \frac{1}{4}$. His model clearly shows that $\frac{1}{4}$ of $\frac{1}{2}$ equals $\frac{1}{8}$. To use models to show $\frac{1}{2} \div \frac{1}{4}$, Alejandro might find it helpful to draw models for both $\frac{1}{2}$ and $\frac{1}{4}$ as shown in Answer Key Figure 10.2.

Answer Key Figure 10.2. Area models representing $\frac{1}{2}$ and $\frac{1}{4}$.



Visually modeling fractions in this manner might help Alejandro determine that two $\frac{1}{4}$s are equal to $\frac{1}{2}$, or $\frac{1}{2} \div \frac{1}{4} = 2$. To see this relationship more clearly, Alejandro could equipartition the model representing halves into fourths.

(b) How might you help Alejandro conceptualize the similarities and differences between division by a fraction and multiplication by a fraction?

Sample Answer

In addition to modeling division by a fraction, in this case $\frac{1}{2} \div \frac{1}{4}$, it is important that Alejandro can conceptualize the difference between multiplying by a fraction and dividing by a fraction. One would want Alejandro to interpret $\frac{1}{2} \div \frac{1}{4}$ as, "How many $\frac{1}{4}$s are there in $\frac{1}{2}$?" Alejandro needs to be able to distinguish this from $\frac{1}{2} \times \frac{1}{4}$, which can be interpreted as $\frac{1}{2}$ of a portion whose size is $\frac{1}{4}$. To reach this understanding, Alejandro should be presented with numerous opportunities to model both fraction multiplication and division problems and to determine which operation is needed to solve a variety of fraction multiplication and division problems.

5. One of Selma's typical responses to division of fractions problems is shown in Figure 10.18. Mr. Latham, Selma's teacher, wants to be sure that Selma possesses the needed conceptual understanding of division of fractions to go along with her algorithmic knowledge.

(a) Based on the evidence in her response, what does Selma appear to understand about division of fractions?

Sample Answer

Selma chose the correct operation for the given problem. She used the "invert and multiply" algorithm correctly and understood that the answer, 8, referred to bags of candy. There is also evidence of Selma's understanding of equivalence. Specifically, she appears to understand that $6 = \frac{6}{1}$ and that $\frac{24}{3} = 8$.

(b) What questions might Mr. Latham ask Selma to help him determine her conceptual understanding of division of fractions?

Sample Answers

Mr. Latham might ask Selma to:

- Construct a model to represent the situation
- Describe another context in which one might compute $6 \div \frac{3}{4}$
- Explain why the answer, 8, is greater than 6, the number of pounds of candy
- Solve the problem using a different strategy
- Explain the steps she used to solve the problem and why the steps she used make sense.

6. Mr. Alberti is preparing for an upcoming lesson on partitive division. As part of the lesson, he plans to use the following problem. Mr. Alberti is contemplating a model that clearly shows the answers to both parts of the question.

Five friends equally share $3\frac{1}{2}$ pizzas.

What fraction of a pizza does each friend get?

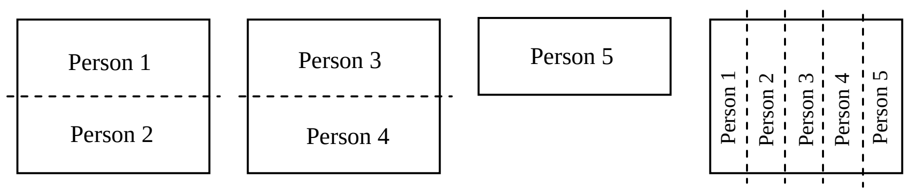
What fraction of all the pizzas does each friend get?

(a) Draw a model clearly showing that each friend gets $\frac{7}{10}$ of a pizza and $\frac{1}{5}$ of the pizzas.

Sample Answer

Answer Key Figure 10.3 is one way to model the problem that each friend gets $\frac{7}{10}$ of a pizza and $\frac{1}{5}$ of the pizzas.

Answer Key Figure 10.3. Sample model showing that each person receives $\frac{1}{5}$ of the pizzas and $\frac{7}{10}$ of a pizza.



Each person receives $\frac{1}{2}+\frac{1}{5}$ or $\frac{7}{10}$ of a pizza. There are 5 equal shares so each person gets $\frac{1}{5}$ of the pizzas.

(b) Explain how you might connect the model you drew for part a to the mathematical calculations $3\frac{1}{2}\div 5$ and 1 ÷ 5.

Sample Answer

The equation $3\frac{1}{2}\div 5$ is represented by the $3\frac{1}{2}$ large rectangles (pizzas) divided into 5 equal shares. Since each person receives $\frac{7}{10}$ of a pizza, the model shows that $3\frac{1}{2}\div 5=\frac{7}{10}$.

To model 1 ÷ 5, one must view the set of $3\frac{1}{2}$ pizzas as the whole. Since each person received $\frac{1}{5}$ of the set of pizzas, $1 \div 5 = \frac{1}{5}$.

7. Cheney's solution to a quotative division problem is shown in Figure 10.37.

(a) What do the numbers on the top of Cheney's number line represent?

Sample Answer

The numbers on the top of Cheney's number line represent the number of yards of wire.

(b) What do the numbers on the bottom of Cheney's number line represent?

Sample Answer

The numbers on the bottom of Cheney's number line represent the number of decorations.

(c) What instructional strategies might you use to help Cheney understand that the fraction $\frac{2}{3}$ represents $\frac{2}{3}$ of a decoration, not $\frac{2}{3}$ of a yard?

Sample Answer

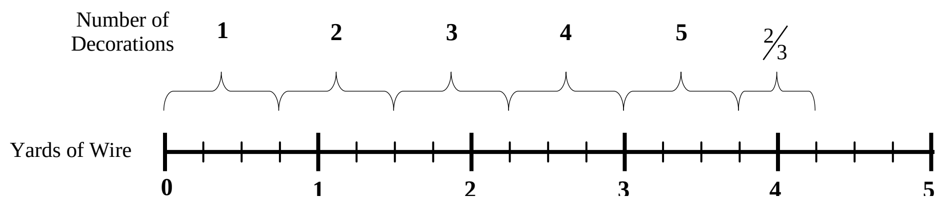
Cheney could benefit from opportunities to describe his model and conceptualize the meaning of number labels on his number line as well as the relationships between the labels (i.e., between the yards of wire and the number of decorations). His bottom labels represent the number of decorations that can be made given $4\frac{1}{4}$ yards of wire. Cheney's model lets him find $5\frac{2}{3}$, meaning that 5 decorations can be made with enough wire remaining for $\frac{2}{3}$ of another decoration. Just as 5 is about the decorations, the $\frac{2}{3}$ in the calculation is about decorations and not yards.

(d) How might you show how this problem results in both $\frac{1}{2}$ of yard and $\frac{2}{3}$ of a decoration left over?

Sample Answer

Although there are many strategies for clarifying the confusion between the $\frac{1}{2}$ yard of wire and $\frac{2}{3}$ of a decoration remaining, all strategies should focus on helping students keep track of the units (i.e., yards of wire and number of decorations) and the relationship between these units. The model in Answer Key Figure 10.4 clarifying Cheney's solution visually shows both of the quantities involved in the problem and the relationship between them. For instance, notice that 3 yards of wire would yield 4 decorations. The model also shows that 5 decorations use $3\frac{3}{4}$ yards of wire. This means that $\frac{1}{2}$ a yard of wire is left over ($4\frac{1}{4}-3\frac{3}{4}=\frac{1}{2}$). In addition, this $\frac{1}{2}$ yard of wire would only make $\frac{2}{3}$ of a decoration because $\frac{1}{2}$ is $\frac{2}{3}$ of $\frac{3}{4}$, the amount of wire it takes for one decoration. This is obviously a difficult concept and students need repeated opportunities to model, discuss, and solve quotative division problems such as this and to interpret and reinterpret remainders.

Answer Key Model 10.4. Model, based on Cheney's solution, showing the relationship between the number of decorations and the yards of wire.



Answer Key Chapter 11: The OGAP Fraction Framework

1) The following addition estimation problem was designed to elicit evidence of student understanding of unit fractions and of the magnitude of fractions.

The sum of $\frac{1}{12}+\frac{7}{8}$ is closest to:

a. 20

b. 8

c. $\frac{1}{2}$

d. 1

Show your work.

Using a copy of the evidence collection sheet in Figure 11.21 analyze the evidence in each of the solutions below.

(a) Based on the evidence:

- **Where on the *OGAP Fraction Progression* is each solution? What is the evidence? Record on the *Evidence Collection Sheet*.**
- **Are there any underlying issues or errors? Record any you identify on the collection sheet.**
- **Is the answer correct? Highlight the names of the students whose answers are incorrect.**

(b) For each piece of student work shown here, identify understandings that can be built upon and a strategy you would use to help the student build understanding toward the next level on the progression.

Sample Answers

Figure 11.17: Noah

- Non-fractional strategy: Used inappropriate whole number reasoning
- Incorrect answer
- Underlying issues or errors: Inappropriate whole number reasoning
- Understandings and instructional strategies: Evidence suggests that Noah understands addition of whole numbers. Use this understanding, together with work with unit fractions, to help Noah build conceptual understanding of addition with simple fractions. Begin with opportunities for Noah to understand that fractions are built from unit fractions. For example, $\frac{3}{5}$ is comprised of three pieces of size $\frac{1}{5}$. Thus: $\frac{3}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$ and $\frac{3}{5}=3\left(\frac{1}{5}\right)$. This concept underlies fraction addition.

Figure 11.18: Jayden

- Transitional strategy: Used a number line to solve the problem
- Correct answer
- Underlying issues or errors: Minor partitioning errors (do not impact the answer)
- Understandings and instructional strategies: Evidence suggests that Jayden can use a number line to solve certain addition problems. Future instruction should include opportunities for Jayden to generalize the understandings she gained through the use of

visual models to develop reasoning strategies based on benchmarks, unit fractions, and reasoning about the relative magnitude of fractions.

Figure 11.19: Emma

- Fractional strategy: Reasoned about the relative magnitude of the two fractions
- Correct answer
- Underlying issues or errors: None
- Understandings and instructional strategies: Evidence suggests Emma can use strategies to reason about the magnitude of fractions. Future opportunities should expand this reasoning to include other operations, such as subtraction and perhaps multiplication and division; other fraction types, such as mixed numbers; fractions greater than one; and negative fractions and problems in context.

Figure 11.20: Ethan

- Transitional strategy: Used visual area models to solve the problem
- Correct answer
- Underlying issues or errors: Equation error: $\frac{7}{8}+\frac{1}{12}$ does not equal 1.
- Understandings and instructional strategies: Ethan used an accurate visual area model to estimate the sum of the two fractions given. Future instruction might include work with number lines, both for locating, ordering and comparing fractions, and for use with estimating sums of fractions. Ultimately, Ethan’s work with visual models will help him develop reasoning strategies for fractions.